Classification of finite spectral triples

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Abstract

It is known that the spin structure on Riemannian manifold can be extended to noncommutative geometry using the notion of spectral triple. For finite geometries, the corresponding finite spectral triples are completely described in terms of matrices and classified using diagrams. When tensorized with the ordinary space–time geometry, finite spectral triples give rise to Yang–Mills theories with spontaneous symmetry breaking, whose characteristic features are given within the diagrammatic approach: vertices of the diagram correspond to gauge multiplets of chiral fermions and links to Yukawa couplings. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

It is nowadays admitted that the major difficulty in quantizing gravity lies in our current conception of geometrical spaces at very short scale. Therefore, a first step towards a quantum theory of gravitation may be a suitable redefinition of the geometrical notions. Due to Heisenberg's uncertainty relations, in a quantum theory, the ordinary notion of a point disappears, so that we have to develop a geometry without any reference to points. This is the basic principle on noncommutative geometry, where points are replaced by the algebra of coordinates, which may be noncommutative. There are many

possible choices of the algebra of coordinates. For example, we can replace the ordinary plane, whose coordinates satisfy $xy = yx$, by the quantum plane, i.e., the algebra generated by the relation $xy = qyx$ for a complex number q. This leads us naturally, when we aim at describing the symmetries of the quantum plane, to the theory of quantum groups.

Another possible choice of the algebra of coordinates lies in replacing it by a suitable involutive subalgebra of the algebra of operators on a Hilbert space. Then, we try to extend, guided by the analogy with quantum mechanics, the ordinary definitions and theorems of differential geometry, to the noncommutative case. This is the framework of *noncommutative geometry* [1–6]. Since matter is fermionic, the notion that should be extended to noncommutative geometry is that of spin structure on a Riemannian manifold. This has been achieved via *spectral triples* in [4,7]. By definition, a spectral triple is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is a usually complex involutive algebra, that stands for the algebra of coordinates, represented on the fermionic Hilbert space $\mathcal{H}$. $\mathcal{D}$ is a self-adjoint operator acting on $\mathcal{H}$, which is the generalization of the Dirac operator. To provide a suitable extension of spin geometry, these three objects are assumed to fulfill some axioms called axioms of noncommutative geometry.

Starting with a suitable class of spectral triples, we can construct gauge theories with spontaneous symmetry breaking. In that case, both the gauge and the Higgs fields appear as connections, the former on an ordinary bundle, the latter on a finite one. Accordingly, the spectral triples we use are products of two spectral triples. The first one encodes all data concerning the geometry of space–time and the second one is what we call a *finite spectral triple*.

In Section 2, we define finite spectral triples and give the relation between the general axioms of noncommutative geometry and the simplest case of finite noncommutative geometry. Section 3 is devoted to a general resolution of the equations we obtained. In Section 4, we introduce some diagrams that allow to classify and to construct finite spectral triples. Finally, in Section 5 we give some applications to model building in particle physics and determine the general feature of the corresponding Yang–Mills theory within the diagrammatic approach.

2. Axioms for finite spectral triples

Finite spectral triples are particular cases of spectral triples of dimension 0. The latter are rigorously defined within the axioms on noncommutative geometry and yield a general theory of discrete spaces. Among all discrete spaces, we focus on finite ones, thus the algebra is finite dimensional. Furthermore, we will also assume that the Hilbert space is finite dimensional, an infinite dimensional one corresponding to a theory with an infinite number of elementary fermions. Accordingly, a finite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is defined as a spectral triple of dimension 0 such that both $\mathcal{A}$ and $\mathcal{H}$ are finite dimensional. Using such a triple, it is possible to construct Yang–Mills theory with spontaneous symmetry breaking whose gauge group is the group of unitary elements of $\mathcal{A}$, $\mathcal{H}$ is the fermionic Hilbert space and $\mathcal{D}$ is the mass matrix.

All finite dimensional involutive algebras that admit involutive representations as oper-taors on Hilbert spaces are direct sum of complex matrix algebras. This remark simplifies considerably finite noncommutative geometry and it becomes possible to give a detailed account of all finite spectral triples. However, to enlarge the spectrum of applications of noncommutative geometry, for instance to include theories with gauge groups $SO(n)$ and $SP(n)$, it is convenient to consider real involutive algebras. The latter are defined as involutive algebras over $\mathbb{R}$ together with a real representation as operators on a Hilbert space. It is known that finite dimensional real involutive algebras which admit a faithful representation on a finite dimensional Hilbert space are just direct sums of matrix algebras over the fields of real numbers, complex numbers and quaternions. Therefore, we write the algebra as a direct sum

$$\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_1}(\mathbb{K}),$$

where $M_n(\mathbb{K})$ denotes the algebra of square matrices of order n with entries in the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. All the results pertaining to spectral triples with complex algebras extend to the real case with only minor modifications. Accordingly, we will focus on complex algebras and only list these modifications in the real case.

In this section, we give the axioms for finite spectral triples and sketch their relations with the general case. Since both the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$, are finite dimensional, all assumptions concerning functional analysis, that are necessary in the general case, are obviously satisfied. Thus, there is a overwhelming simplification in the formulation of noncommutative geometry and the seven axioms involved in [7] reduce to the four following ones: *orientability, reality, Poincaré duality* and *first order condition*. To proceed, we fix a finite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ together with a faithful and unitary representation π of $\mathcal{A}$ as operators on $\mathcal{H}$. Moreover, the formulation of the axioms of noncommutative geometry involves two other operators, the *charge conjugation* $\mathcal{J}$ and the *chirality* χ that we shall also fix. The precise definition of these two operators will become clearer in the following sections.

2.1. Orientability

Since the noncommutative manifold defined by a spectral triple has to be oriented, we must specify an orientation form. In the case of an ordinary manifold of dimension n, this is acheived by means of a nowhere vanishing n-form γ. The extension of this notion to the noncommutative case goes through the deep relation, in the commutative case, between differential forms and Hochschild homology.

Let us now briefly recall some elementary aspects of Hochschild homology, referring to [10] for a general theory. Given an associative, unital, but not necessarily commutative algebra $\mathcal{A}$, and a bimodule $\mathcal{M}$ over $\mathcal{A}$, we first define, for any integer n, the vector space $T_n(\mathcal{A}, \mathcal{M})$ as the tensor product of $\mathcal{M}$ by n copies of $\mathcal{A}$

$$T_n(\mathcal{A}, \mathcal{M}) = \mathcal{M} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}.$$

Then, we define, for any $n > 0$, the boundary operation b_n as a linear map from $T_n(\mathcal{A}, \mathcal{M})$ to $T_{n-1}(\mathcal{A}, \mathcal{M})$ such that

$$
\begin{aligned}
b_n(m &\otimes x_1 \otimes \cdots \otimes x_n) \\
&= m x_1 \otimes x_2 \otimes \cdots \otimes x_n + (-1)^n x_n m \otimes x_1 \otimes \cdots \otimes x_{n-1} \\
&\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes x_1 \otimes \cdots x_i x_{i+1} \otimes \cdots \otimes x_n.
\end{aligned}
$$

For $n = 0$ we set $b_0 = 0$. By definition, the Hochschild cycles $Z_n(\mathcal{A}, \mathcal{M})$ of dimensional n are the elements of $T_n(\mathcal{A}, \mathcal{M})$ whose image under b_n is zero. Now, let us consider the bimodule $\mathcal{M} = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$, whose bimodule structure is given by

$$
y(x \otimes x') y' = yxy' \otimes x'
$$

for any $y, y' \in \mathcal{A}$ and $x \otimes x' \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

These rather abstract objects are represented as operators on the Hilbert space. Using the Dirac operator $\mathcal{D}$, the representation π can be extended to the Hochschild cycles as

$$
\begin{aligned}
\pi(x &\otimes x' \otimes y_1 \otimes \cdots \otimes y_n) \\
&= \pi(x) \mathcal{J} \pi(x') \mathcal{J}^{-1} [\mathcal{D}, \pi(y_1)] \cdots [\mathcal{D}, \pi(y_n)]
\end{aligned}
$$

for any $x \otimes x' \otimes y_1 \otimes \cdots \otimes y_n \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$. The previous representation of Hochschild cycles involves an antiunitary operator $\mathcal{J}$. The latter is nothing but the charge conjugation that will be defined below. For the time being, we shall simply consider it as a given antiunitary operator since we will not use any if its specific properties. This leads to a natural extension of the orientability of a manifold to the noncommutative case [7].

Orientability. For a noncommutative geometry of dimension n (n even), the chirality χ is a hermitian involution that commutes with $\pi(x)$ for any $x \in \mathcal{A}$, anticommutes with the Dirac operator $\mathcal{D}$ and is the image of an n-dimensional Hochschild cycle.

Before we focus on finite spectral triples, let us check that it holds for the ordinary Euclidean geometry of space–time. The latter is a compact four dimensional manifold V, the algebra $\mathcal{A}$ is the algebra $C^{\infty}(V)$ of cimplex valued smooth functions on V, the Hilbert space $\mathcal{H}$ is the space $L^2(V, S)$ of square integrable sections of the spinor bundle of V, D is the ordinary Dirac operator defined in a local chart by $\mathcal{D} = i\gamma^\mu \partial_\mu$, γ^μ being the usual Euclidean Dirac matrices, and $\mathcal{J}$ is the charge conjugation of Dirac spinors. The functions act simply on $\mathcal{H}$ by multiplication and we have, for any $f \in C^{\infty}(V)$,

$$
[\mathcal{D}, \pi(f)] = i\gamma^\mu \partial_\mu f.
$$

Since the algebra is commutative, the bimodule $\mathcal{M}$ involved in the definition of Hochschild homology is not relevant and we simply take the Hochschild cycles $Z_n(\mathcal{A}, \mathcal{M})$ with $\mathcal{M} = \mathcal{A}$. Accordingly, any 0-form is function and can be written as the image of a zero dimensional

cycle. Moreover, Hochschild cycles of dimension 1 give us 1-forms. Indeed, starting with the cycle $\sum_i f_0^i \otimes f_1^i$, its image is

$$\sum_i \mathrm{i} f_0^i \partial_\mu f_1^i \gamma^\mu,$$

that can be considered as the 1-form $A_\mu \, \mathrm{d}x^\mu$ if we identify $\sum_i f_0^i \partial_\mu f_1^i$ with A_μ and $\mathrm{i}\gamma^\mu$ with $\mathrm{d}x^\mu$.

At higher orders, the situation is more subtle but we can still construct, using antisymmetrization, some Hochschild cycles whose images are differentials forms if we identify $\mathrm{i}\gamma_\mu$ with $\mathrm{d}x^\mu$. In fact, for any $f_0, f_1, \ldots, f_n \in C^\infty(V)$, we define

$$c = \sum_\sigma \epsilon(\sigma) f_0 \otimes f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

where the sum runs over all permutations of $1, 2, \ldots, n$ and $\epsilon(\sigma)$ is the signature of σ. Due to the commutativity of $C^\infty(V)$, we check that c is a Hochschild cycle and that, thanks to the antisymmetrization procedure, its image is an n-form. This applies for the volume form that can be defined as

$$\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma.$$

If we identify $\mathrm{d}x^\mu$ with $\mathrm{i}\gamma^\mu$, the previous relation is the definition of γ^5 so that γ^5 turns out to be the chirality, and it corresponds to the volume form.

Let us give two remarks pertaining to the commutative case. First, the antisymmetrization procedure yields us immediately differential forms, so that we do not have to divide by some auxiliary fields (the so-called) "junk" defined in [3]. Secondly, we have seen that Hochschild cycles provide us with differential forms when represented as operators on the Hilbert space. Thus, it is natural to ask for an exterior derivative in this framework. More precisely, we would like to have some linear maps d_n from $Z_n(\mathcal{A}, \mathcal{A})$ into $Z_{n+1}(\mathcal{A}, \mathcal{A})$ such that $d_{n+1} \circ d_n = 0$ and such that, when represented with Dirac matrices, they coincide with the genuine exterior derivatives. In fact, we can set $d_n = B_n$, where B_n is Connes' coboundary map [10]. For an associative and unital algebra $\mathcal{A}$, the latter is defined as a linear map from $T_n(\mathcal{A}, \mathcal{A})$ into $T_{n+1}(\mathcal{A}, \mathcal{A})$ by

$$
\begin{aligned}
&B_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n) \\
&= \sum_{i=0}^{n-1} (-1)^{ni} 1 \otimes x_i \otimes \cdots \otimes x_n \otimes x_0 \otimes \cdots \otimes x_{i-1} \\
&\quad - \sum_{i=0}^{n-1} (-1)^{n(i-1)} x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n \otimes x_0 \otimes \cdots \otimes x_{i-2}
\end{aligned}
$$

for any $x_0, \ldots, x_n \in \mathcal{A}$. It is easy to check that $B_{n-1} \circ b_n + b_{n+1} \circ B_n = 0$. Therefore, $B_n(Z_n(\mathcal{A}, \mathcal{A}))$ is included in $Z_{n+1}(\mathcal{A}, \mathcal{A})$. Moreover, one has $B_{n+1} \circ B_n = 0$ and thus B_n can be considered as an exterior derivative in the framework of Hochschild homology. Besides, in the commutative case, when we lift this coboundary operation on operators on

the Hilbert space using Dirac matrices, it yields the genuine exterior derivative without introducing auxiliary fields. Let us check it for 1-forms. If $A_\mu \, dx^\mu$ is a 1-form, it can be written as the image of the Hochschild cycle $\sum_i f_0^i \otimes f_1^i$, with $A_\mu = \sum_i f_0^i \partial_\mu f_1^i$. The action B_1 on this cycle is given by

$$
B_1 \left(\sum_i f_0^i \otimes f_1^i \right)
$$
$$
= \sum_i (1 \otimes f_0^i \otimes f_1^i - 1 \otimes f_1^i \otimes f_0^i + f_1^i \otimes 1 f_1^i - f_1^i \otimes 1 \otimes f_0^i).
$$

When we represent it, we end up with

$$
i\gamma_\mu i\gamma_\nu (\partial_\mu f_0^i \partial_\nu f_1^i - \partial_\mu f_1^i \partial_\nu f_0^i)
$$
$$
= i\gamma_\mu i\gamma_\nu (\partial_\mu A_\nu - \partial_\mu A_\nu) = F_{\mu\nu} \, dx^\mu \wedge dx^\nu
$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$ and if we identify $i\gamma^\mu$ with dx^μ. Let us note that this also works for higher-order forms.

For finite spectral triples, the orientability axiom requires that χ must be written as the image of a Hochschild cycle of dimension 0. Since all elements of $T_0(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$ are cycles, it only states that there are some elements $a_i, b_i \in \mathcal{A}$ such that $\chi = \sum_i \pi(a_i) \mathcal{J} \pi(b_i) \mathcal{J}^{-1}$. Accordingly, the orientability axiom is reformulated as follows:

Orientability (finite case). The chirality χ is an hermitian involution that commutes with $\pi(x)$ for any $x \in \mathcal{A}$, anticommutes with $\mathcal{D}$ and can be written as $\chi = \sum_i \pi(x_i) \mathcal{J} \pi(y_i) \mathcal{J}^{-1}$ for some $x_i, y_i \in \mathcal{A}$.

For the standard model, χ is simply the operator of $\mathcal{H}$ that takes the value -1 for left-handed fermions $+1$ for right-handed fermions.

2.2. Reality

To define noncommutative manifolds, we need, in this setting, a suitable notion of Poincaré duality. A first step towards a formulation of Poincaré duality in noncommutative geometry lies in introducing a bimodule structure on $\mathcal{H}$. Note that the bimodule structure is also necessary if one wants to incorporate the color sector in the Connes–Lott model [3]. However, this bimodule is not arbitrary and it has to fulfill the reality axiom.

Reality (finite case). There is an antiunitary involution $\mathcal{J}$ on $\mathcal{H}$ (i.e., $\mathcal{J}^* = \mathcal{J}^{-1} = \mathcal{J}$) that commutes with χ such that for any $x, y, \in \mathcal{A}$, one has $[\pi(x), \mathcal{J}\pi(y)\mathcal{J}^{-1}] = 0$.

Then we can equip $\mathcal{H}$ with a bimodule structure. The left action is

$$
(x, \psi) \in \mathcal{A} \times \mathcal{H} \mapsto x\psi = \pi(x)\psi,
$$

and the right action is

$$
(\psi, y) \in \mathcal{H} \times \mathcal{A} \mapsto \mathcal{J}\pi(y^*)\mathcal{J}^{-1}\psi.
$$

This defines an $(\mathcal{A}, \mathcal{A})$-bimodule structure or equivalently, a $(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$-module structure on $\mathcal{H}$. Note that at a mathematical level, the introduction of this operator has been motivated by the transition from complex K-theory to real K-theory that turns out to be necessary in the formulation of noncommutative spin geometry. Moreover, given a cyclic and separating vector in $\mathcal{H}$, $\mathcal{J}$ is Tomita–Takesaki involution that allows us to pass from $\mathcal{A}$ to its commutant $\mathcal{A}'$. For the standard model, this operator is just charge conjugation. In other words, the Hilbert space of the finite spectral triple of the standard model is the vector space spanned by all fermions and antifermions and the operator $\mathcal{J}$ is the antilinear operator a fermion to the corresponding antifermion.

We gave the reality axiom in the finite case only. For general spectral triples, we only have to change some signs in the relations involving $\mathcal{J}$, depending on the *dimension* modulo 8 of the spectral triple [7]. In the commutative case of a smooth manifold V endowed with a spin structure, $\mathcal{J}$ is the charge conjugation defined on the space of square integrable sections of the spinor bundle of a manifold V.

2.3. Poincaré duality

For a closed Riemannian manifold V of dimension n, there is a well-known isomorphism between the de Rham groups $H^p(V)$ and $H^{n-p}(V)$ given by the bilinear map

$$(f, g) \mapsto \int_V f \wedge *g,$$

where f and g are closed p and $(n-p)$-forms that are representatives of the de Rham groups $H^p(V)$ and $H^{n-p}(V)$. Formulating this in noncommutative geometry is a rather difficult task that requires the use of Kasparov's biinvariant theory. We refer to [2,7] for a detailed account of Poincaré duality in noncommutative geometry. In this section, we simply give the axiom of Poincaré duality, as stated in [7], and derive its formulation in the finite case. The Poincaré duality axiom is the following.

Poincaré duality . The intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ is nondegenerate.

Some insights on V are given by the knowledge of the vector bundles over V through its K-theory that can be formulated algebraically. $K_*(\mathcal{A})$ is a collective notation for all K-groups. Due to Bott periodicity, the sequence of these groups is periodic with period 2, for complex algebras. Moreover in the case of matrix algebras, the group $K_1(\mathcal{A})$ vanishes identically, so that we are left with the study of $K_0(\mathcal{A})$ only. In the case of real algebras, the period is 8 [11]. However, if we do not take into account the torsion (i.e. the presence of K-groups of type $\mathbb{Z}/2\mathbb{Z}$), only matters the group $K_0(\mathcal{A})$. Since K_0 is additive, i.e.,

$$K_0 \left(\bigoplus_{i=1}^N M_{n_i}(\mathbb{K}) \right) = \bigoplus_{i=1}^N K_0(M_{n_i}(\mathbb{K})),$$

we have to compute $K_0(M_n(\mathbb{K}))$ [12]. By definition, $K_0(M_n(\mathbb{K}))$ is the Grothendieck group of stably isomorphic classes of finitely generated projective modules over $\mathcal{A}$. Recall that a

module over $M_n(\mathbb{K})$ is a vector space E equipped with a representation of $M_n(\mathbb{K})$ which is said to be finitely generated if any of its elements can be written as a linear combination with coefficients in $M_n(\mathbb{K})$ of a finite number of elements of E. The additive structure is given by the direct sum of modules, and two such modules are said to be stably isomorphic if they are isomorphic after addition of a trivial module (i.e., a module that is simply a direct sum of reversal copies of the algebra).

The direct sum of modules yielding only an addition, we have to define the opposite of a module to obtain a group. This is achieved by means of the Grothendick procedure, which is used to construct the additive group of integers $\mathbb{Z}$ out of the positive integers $\mathbb{N}$. Thus, we end up with an Abelian groups, which turns out to be the free group generated by the class of the "smallest" modules over $M_n(\mathbb{K})$. The latter is given by any nonzero projection in $M_n(\mathbb{K})$ of minimal rank, since two projections of same rank are obviously equivalent and since the rank of the direct sum of two projections is the sum of their ranks. Now, we simply have to find such a projection in the real, complex and quaternionic cases. In the first two cases, the solution is obvious since any matrix whose only nonvanishing entry on the diagonal is equal to 1 is a projection of minimal rank. Therefore such a projection has trace 1. In the quaternionic case, it remains true if we consider that only nonvanishing entry is the unity of $\mathbb{H}$, so that it has trace 2. Finally, $K_0(\mathcal{A})$ turns out to be $\mathbb{Z}^N$ and is the free group generated by $(p_i)_{1 \leq i \leq N}$, where $p_i \in M_{n_i}(\mathbb{K})$ is a self-adjoint projection of minimal rank. As an additive group, it can be considered as a $\mathbb{Z}$-module with basis $(p_i)_{1 \leq i \leq N}$.

To formulate Poincaré duality for finite spectral triples, we have to compute the intersection form $\cap$ in this case. More precisely, since $\cap$ is $\mathbb{Z}$-bilinear form on $K_0(\mathcal{A}) \times K_0(\mathcal{A})$, we have to compute its matrix given by $\cap_{ij} = \cap(p_i, p_j)$. To proceed, we apply the general definition of the intersection form as the pairing of the Dirac operator index with the two projections p_i and p_j. Finite spectral triples are even spectral triples, so that we rewrite the Dirac operator as

$$\mathcal{D} = M + M^*,$$

where

$$M = \tfrac{1}{4}(1 - \chi)\mathcal{D}(1 + \chi).$$

M is a map from the space of right-handed fermions (eigenspace of χ associated to the eigenvalue $+1$) to the space of left-handed fermions (eigenspace of χ associated to the eigenvalue -1), and can be considered as a mass matrix. Using the reality axiom, the Hilbert space is endowed with a $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$-module structure that follows us to the linear map

$$M_{ij} = \pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}M\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}.$$

In fact, M_{ij} is a map from $\mathcal{H}_{ij}^{\mathrm{R}}$ into H_{ij}^{L}, where

$$\mathcal{H}_{ij}^{\mathrm{R}} = \tfrac{1}{2}\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}(1 + \chi)\mathcal{H},$$
$$\mathcal{H}_{ij}^{\mathrm{L}} = \tfrac{1}{2}\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}(1 - \chi)\mathcal{H}.$$

Both spaces being finite dimensional, the index is easily computed and we get

$$\text{Index}(M_{ij}) = \dim(\mathcal{H}_{ij}^{R}) - \dim(\mathcal{H}_{ij}^{L}).$$

Since χ commutes with $\pi(p_i)$ and $\mathcal{J}\pi(p_j)\mathcal{J}^{-1}$, the operators $\frac{1}{2}\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}(1 + \chi)$ and $\frac{1}{2}\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}(1 - \chi)$ are projections. Therefore, the dimension of $\mathcal{H}_{ij}^{R}$ and $\mathcal{H}_{ij}^{L}$ are simply the trace of the corresponding projections. Finally, $\cap_{ij}$, defined as the pairing of $p_i \otimes p_j$ with the index of $\mathcal{D}$, is given by

$$
\begin{aligned}
\cap_{ij} &= \text{Index}(M_{ij}) \\
&= \dim(\mathcal{H}_{ij}^{R}) - \dim(\mathcal{H}_{ij}^{L}) \\
&= \tfrac{1}{2}\text{Tr}[\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}(1 + \chi)] - \tfrac{1}{2}\text{Tr}[\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}(1 - \chi)] \\
&= \text{Tr}[\chi(\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1})].
\end{aligned}
$$

Accordingly, we have obtained a simple formulation of Poincaré duality for the finite spectral triples that can be stated as follows.

Poincaré duality (finite case). The matrix defined by $\cap_{ij} = \text{Tr}[\chi(\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1})]$ has nonvanishing determinant, where $p_i \in M_{n_i}(\mathbb{K})$ is a self-adjoint projection of minimal rank.

2.4. First-order condition

In ordinary differential geometry of a compact manifold endowed with a spin structure, the Dirac operator is a first-order differential operator. This can be formulated using commutation relations. Indeed, let V, be a compact manifold and E and F be smooth vector bundles over V. The space of sections $\Gamma(E)$ and $\Gamma(F)$ of the vector bundles turns out to be modules over $C^\infty(V)$ and one can easily characterize first-order operators. They are the differential operators $D : \Gamma(E) \to \Gamma(F)$ that fulfill

$$[[D, f], g] = 0$$

for any couple of smooth functions $f, g, \in C^\infty(V)$. This can also be extended to higher-order differential operator $\mathcal{D}$ is of order n if only if, for any $f \in C^\infty(V)[\mathcal{D}, f]$ is of order $n - 1$.

However, this notation cannot be extended in direct manner in the noncommutative setting, In fact, we have to replace the modules by bimodules, and the functions in the double commutator by the left and right actions of the algebras. Since the latter commute, it does not matter whether the left or the right action appears first.

First-order condition. $[[\mathcal{D}, \pi(x)], \mathcal{J}\pi(y)\mathcal{J}^{-1}] = 0$ for any $x, y \in \mathcal{A}$.

We can rewrite the previous relation using bimodule notations. It becomes

$$\mathcal{D}(x\psi y) = x\mathcal{D}(\psi y) + \mathcal{D}(x\psi)y - x\mathcal{D}(\psi)y$$

for any $\psi \in \mathcal{H}$ and $x, y \in \mathcal{A}$. This is the usual definition of a first-order operator on a bimodule [13].

2.5. S^0-reality

The last four axioms define finite spectral triples. However, it turns out that the spectral triple of the standard model belongs to the smaller class of finite spectral triples that exclude Majorana particles. The operator $\mathcal{J}$ can be considered as a charge conjugation operator and it is natural to ask whether there are Majorana fermions. Spectral triples without Majorana fermions, such as the spectral triple of the standard model, are called S^0-real spectral triples. They are defined by imposing a fifth axiom called.

S^0-*reality*. There is a hermitian involution γ that commutes with χ, $\mathcal{D}$ and $\pi(x)$ for any $x \in \mathcal{A}$ and anti-commutes with $\mathcal{J}$.

Since γ is a hermitian involution, one has $\gamma^* = \gamma$ and $\gamma^2 = 1$, so that one can decompose $\mathcal{H}$ into a direct sum of eigenspaces of γ. The eigenspace corresponding to eigenvalue $+1$ (resp. -1) is spanned by particles (resp. antiparticles). The anticommutation rule between γ and $\mathcal{J}$ teaches us that there is no particle that is its own antiparticle. In that case, $\mathcal{H}$ is a direct sum,

$$\mathcal{H} = \mathcal{H}_L^P \oplus \mathcal{H}_R^P \oplus \mathcal{H}_L^A \oplus \mathcal{H}_R^A,$$

where P and A, refer to particle and antiparticle and L and R to left and right. Within this decomposition, π and χ are block diagonal,

$$\pi = \text{diag}(\pi_L^P, \pi_R^P, \pi_L^A, \pi_R^A),$$
$$\chi = \text{diag}(-I_{n_L}, I_{n_R}, -I_{n_L} I_{n_R}),$$

with $n_L = \dim(\mathcal{H}_L^P) = \dim(\mathcal{H}_L^A)$ and $n_R = \dim(\mathcal{H}_R^P) = \dim(\mathcal{H}_R^A)$. In a suitable basis, $\mathcal{J}$ and $\mathcal{D}$ are given by

$$\mathcal{D} = \begin{pmatrix} 0 & M & 0 & 0 \\ M^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{M} \\ 0 & 0 & \overline{M} & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 0 & I_{n_L} & 0 \\ 0 & 0 & 0 & I_{n_R} \\ I_{n_L} & 0 & 0 & 0 \\ 0 & I_{n_R} & 0 & 0 \end{pmatrix} C,$$

where C denotes the complex conjugation and $M \in M_{n_L \times n_R}(\mathbb{C})$ is a mass matrix. The reality and first-order axioms are simply

$$[\pi_L^P(x), \overline{\pi}_L^A(y)] = 0, \qquad [\pi_R^P(x), \overline{\pi}_R^A] = 0,$$
$$(M\pi_R^P)(x) - \pi_L^P(x)M)\overline{\pi}_R^A(y) = \overline{\pi}_L^A(y)(M\pi_R^P(x) - \pi_L^P(x)M),$$

for any $x, y, \in \mathcal{A}$. Accordingly, particle and antiparticle spaces yield two bimodules, the latter being the opposite of the former in the sense that we exchange, up to complex conjugation, the left and right actions. Moreover, the Dirac operator, when restricted to the particle bimodule, is a first-order operator, and it acts on antiparticle space as the complex conjugate. Therefore, the study of S^0-real spectral triple can be reduced to study of the particle bimodule.

3. General solution

The axioms of noncommutative turn out to be very simple in the finite case. Moreover, the algebra $\mathcal{A}$ is a direct sum of matrix algebras and it becomes possible to give a general solution to these axioms.

3.1. The representation

We first assume that the algebra is made of complex matrices with a complex representation, i.e., a representation linear over the complex numbers, and then, we extend our results to the real case. Let $\mathcal{A}$ be a direct sum of complex matrix algebras,

$$\mathcal{A} = \bigotimes_{i=1}^{N} M_{n_i}(\mathbb{C}).$$

To fix notations, we write elements x, y of the algebra as N-uples: $x = (x_i)_{1 \leq i \leq N}$, with $x_i \in M_{n_i}(\mathbb{C})$. The only irreducible representation of $M_n(\mathbb{C})$ as a complex associative algebra is the fundamental one, so that π can be reduced as

$$\pi(x) = \bigotimes_i \bigotimes I_{m_i},$$

where, for $m \in \mathbb{N}$, I_m denotes the identity of $M_m(\mathbb{C})$. If $m = 0$, we assume that I_m is equal to zero and does not appear in the previous decomposition.

The reality axiom provides us with two commuting representations of $\mathcal{A}$, $x \mapsto \pi(x)$ and $y \mapsto \mathcal{J}\pi(\bar{y})\mathcal{J}^{-1}$, that can be simultaneously decomposed into irreducible ones. We end up with a *matrix of multiplicities*, $m \in M_N(\mathbb{N})$, and we have

$$\pi(x) = \bigotimes_{ij} x_i \otimes I_{m_{ij}} \otimes I_{n_j},$$

$$\mathcal{J}\pi(y)\mathcal{J}^{-1} = \bigotimes_{ij} I_{n_i} \otimes I_{m_{ij}} \otimes \bar{y}_j.$$

To this reduction of π is associated the corresponding decomposition of $\mathcal{H}$ as a direct sum of mutually orthogonal subspaces,

$$\mathcal{H} = \bigotimes_{ij} \mathcal{H}_{ij},$$

where $\mathcal{H}_{ij}$ is isomorphic to $\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_{ij}} \otimes \mathbb{C}^{n_j}$. The action of $\pi(x)$ and $\mathcal{J}\pi(y)\mathcal{J}^{-1}$ on $\mathcal{H}_{ij}$ are $x_i \otimes I_{m_{ij}} \otimes I_{n_j}$ and $I_{n_i} \otimes I_{m_{ij}} \otimes \bar{y}_j$, respectively. Accordingly, $\mathcal{H}_{ij}$ can be identified, as a bimodule, with m_{ij} copies of $M_{n_i \times n_j}(\mathbb{C})$, where the left and right actions are simply left and right matrix multiplications.

In the real case, the situation is a bit more tricky. The algebras $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ have only one irreducible representation, the fundamental one, whereas $M_n(\mathbb{C})$ has two inequivalent irreducible representations, the fundamental one and its complex conjugate. However, all our

formulae remain valid, provided the indices i and j label irreducible representations of $\mathcal{A}$, In this case, x_i denotes the image of $x \in \mathcal{A}$ by the irreducible representation corresponding to i, and $\mathcal{H}_{ij}$ is isomorphic to $\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_{ij}} \otimes \mathbb{C}^{n_j}$, where n_i is the dimension of the irreducible representation i.

To introduce the matrix of multiplicities m, we only use the bimodule structure of $\mathcal{H}$. Thus, to any $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$, where $\mathcal{M}$ is finite dimensional and $\mathcal{A}$ and $\mathcal{B}$ are direct sums of matrix algebras, we associate a matrix of multiplicity m by reducing simultaneously the left and right actions. The row index of m labels the irreducible representations of $\mathcal{A}$ whereas its column index labels those of $\mathcal{B}$. Furthermore, it is possible to perform some algebraic operations on the matrices of multiplicities:

- If $\mathcal{M}$ is a $(\mathcal{A}, \mathcal{B})$-bimodule with matrix m, the opposite $(\mathcal{B}, \mathcal{A})$-bimodule corresponds to m^*.
- If $\mathcal{M}$ and $\mathcal{N}$ are $(\mathcal{A}, \mathcal{B})$-bimodules with matrices m and n, the direct sum $\mathcal{M} \oplus \mathcal{N}$ is a $(\mathcal{A}, \mathcal{B})$-bimodule associated to the matrix $m + n$.
- If $\mathcal{M}$ is a $(\mathcal{A}, \mathcal{B})$-bimodule with matrix m and $\mathcal{N}$ a $(\mathcal{B}, \mathcal{C})$-bimodule with matrix n, the tensor product $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$ is a $(\mathcal{A}, \mathcal{C})$-bimodule that corresponds to product mn.

This is very simple case of the theory of composition of correspondences, where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are general Von Neumann algebras [4,14]. It applies in particular for a S^0-real spectral triple. The particle space is a bimodule associated to a matrix e, the antiparticle space is associated to e^* and the total space corresponds to $m = e + e^*$, so that the matrix of multiplicities is symmetric.

Conclusion. A bimodule over a couple of semisimple involutive algebras is given, up to unitary equivalence, by a matrix of multiplicities whose entries are positive integers and whose row and column indices label irreducible representations of the corresponding algebras.

3.2. The charge conjugation

Given a $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{H}$ with matrix m, it is natural to ask whether it corresponds to a real structure (i.e., we can pass from one representation to the other by means of a charge conjugation $\mathcal{J}$ as above) or not. Furthermore, if $\mathcal{J}$ exists, it is interesting to determine its most general form.

To answer these questions, let us first assume that if exists. The action of $\mathcal{J}$ on $\psi \in \mathcal{H}$ can always be written, in any basis, as $\mathcal{J}\psi = K\overline{\psi}$, where K is a matrix. $\mathcal{J}$ is an antiunitary involution if only if $K\overline{K} = KK^* = 1$. We must also have

$$K \left(\bigotimes_{i,j} x_i \otimes I_{m_{ij}} \otimes I_{n_j} \right) = \left(\bigotimes_{i,j} I_{n_i} \otimes I_{m_{ij}} \otimes x_j \right) K$$

for any $x \in \mathcal{A}$. Accordingly,

$$K \left(\bigotimes_j \mathcal{H}_{ij} \right) \subset \bigotimes_j \mathcal{H}_{ji},$$

and since $K^{-1} = K^*$,

$$K(\mathcal{H}_{ij}) = \mathcal{H}_{ji}.$$

Therefore, the dimensions of $\mathcal{H}_{ij}$ and $\mathcal{H}_{ji}$ are equal, or equivalently, the matrix of multiplicities m is symmetric: $m_{ij} = m_{ji}$.

Let us denote by $K_{ij} : \mathcal{H}_{ij} \to \mathcal{H}_{ji}$ the restriction of K. It must fulfill

$$K_{ij}(x_i \otimes I_{m_{ij}} \otimes I_{n_j}) = (I_{n_i} \otimes I_{m_{ij}} \otimes \times x_j)K_{ij},$$
$$K_{ij}K_{ij}^* = K_{ij}\overline{K_{ij}} = 1.$$

Consequently, if $e_i^a \otimes \psi_{ij}^p \otimes e_j^b$ is a basis of $\mathcal{H}_{ij} = \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_{ij}} \otimes \mathbb{C}^{n_j}$, we have

$$K_{ij}(e_i^a \otimes \psi_{ij}^p \otimes e_j^p) = e_j^b \otimes L_{ij}\psi_{ij}^p \otimes e_i^a,$$
$$L_{ij}L_{ij}^* = 1.$$

If $i \neq j$, the spaces $\mathcal{H}_{ij}$ and $\mathcal{H}_{ji}$ are distinct and it is always possible, by a suitable choice of the orthonormal basis $(\psi_{ij}^p)_{1 \leq p \leq m_{ij}}$, to have $L_{ij}\psi_{ij}^p = \psi_{ji}^p$. When $i = j$, L_{ii} is unitary, and we can choose an orthonormal basis such that $L_{ii}\psi_{ii}^p = e^{i\phi_i^p}\psi_{ii}^p$, with $\phi_i^p \in \mathbb{R}$. However, since $\mathcal{J}$ is antilinear,

$$\mathcal{J}(e^{i\phi_i^p/2}e_i^a \otimes \psi_{ii}^p \otimes e_i^b) = e^{i\psi_i^p/2}K(\overline{e_i^a} \otimes \overline{\psi_{ii}^p} \otimes \overline{e_i^b}),$$
$$= e^{-i\phi_i^p/2}\overline{e_i^a} \otimes L_{ii}\overline{\psi_{ii}^p} \otimes \overline{e_i^b},$$
$$= e^{+i\phi_i^p/2}\overline{e_i^a} \otimes \overline{\psi_{ii}^p} \otimes \overline{e_i^b},$$

so that the phases disappear if we make the replacement $\psi_{ij}^p \to e^{i\phi_i^p/2}\psi_{ij}^p$. Thus the action of $\mathcal{J}$ is simply

$$\mathcal{J}(e_i^a \otimes \psi_{ij}^p \otimes e_j^b) = \overline{e_j^b} \otimes \overline{\psi_{ji}^p} \otimes \overline{e_i^a}$$

in all cases.

Conversely, if the matrix of multiplicities m is symmetric, it is possible to define $\mathcal{J}$ as above and such an antilinear operator fulfills all conditions imposed to charge conjugation.

In the S^0-real case, we have already proved that $m = e + e^*$. In this case, we have two bimodules, one with matrix e and the other with matrix e^*. If we define $\mathcal{J}$ as the antilinear map that exchange the corresponding vectors of the two bimodules and γ the hermitian operator that takes value 1 on the first bimodule and -1 on the second one, they anticommute and the direct sum of the two bimodules corresponds, at the bimodule level (i.e., if we do not take care of χ and $\mathcal{D}$), to a S^0-real structure.

Conclusion. A $(\mathcal{A}, \mathcal{A})$-bimodule with matrix of multiplicities m is endowed with a real structure if only if m is symmetric. Moreover, if so, the charge conjugation is unique up to unitary equivalence, and it acts as above on tensor products. It corresponds to a S^0-real structure if only if m can be written as $m = e + e^*$, with $e \in M_N(\mathbb{N})$, or equivalently, if all diagonal entries of m are even.

3.3. The chirality

The chirality χ commutes with $\pi(\mathcal{A})$ and $\mathcal{J}\pi(\mathcal{A})\mathcal{J}^{-1}$, and is hermitian, so that it can be written as a block diagonal matrix,

$$\chi = \bigotimes_{ij} I_{n_i} \otimes \chi_{ij} \otimes I_{n_j}.$$

Since $\chi = \sum_p \pi(x^p)\mathcal{J}\pi(y^p)\mathcal{J}^{-1}$, with $x^p, y^p \in \mathcal{A}$, χ_{ij} is scalar matrix. Furthermore $\chi_{ij} = \pm 1$ as follows from $\chi^2 = 1$. We also have $[\chi, \mathcal{J}] = 0$ so that $\chi_{ij} = \chi_{ji}$.

All the information contained in the matrices χ_{ij} and m_{ij} can be cast in a symmetric matrix $\mu \in M_N(\mathbb{Z})$ defined by

$$\mu_{ij} = \chi_{ij} m_{ij}.$$

We recover χ_{ij} and m_{ij} as the sign and the modulus of μ_{ij}. Accordingly,

$$\chi = \bigotimes_{ij} \text{sign}(\mu_{ij}) I_{n_i} \otimes I_{|\mu_{ij}|} \otimes I_{n_j}.$$

To proceed further, let us assume that we are dealing eith the complex case. Thus all projections involved in Poincaré duality are of trace 1, and the matrix of the intersection form $\cap_{ij}$ is given by

$$\begin{aligned}
\cap_{ij} &= \text{Tr}[\chi\pi(p_i)\mathcal{J}\pi(p_j)\mathcal{J}^{-1}], \\
&= \text{sign}(\mu_{ij})\,\text{Tr}[p_i \otimes I_{|\mu_{ij}|} \otimes \overline{p}_j], \\
&= \mu_{ij}.
\end{aligned}$$

According to Poincaré duality, the matrix μ must be nondegenerate.

Conversely, if we start with a symmetric and nondegenerate matrix $\mu \in M_N(\mathbb{Z})$ (from now on we refer to it as the matrix of multiplicities instead of m), we recover the bimodule structure out of $|\mu_{ij}|$ and the chirality χ is obtained as

$$\chi = \bigotimes_{ij} \text{sign}(\mu_{ij}) I_{n_i} \otimes I_{|\mu_{ij}|} \otimes I_{n_j}.$$

It fulfills all the requirements imposed by Poincaré duality and by the orientability axiom. For instance, let us check that there are some elements, x^p and y^p, of $\mathcal{A}$ such that $\chi = \sum_p \pi(x^p)\mathcal{J}\pi(y^p)\mathcal{J}^{-1}$. To proceed, let us expand the matrix $\chi_{ij} = \text{sign}(\mu_{ij})$ over rank one matrices as

$$\chi_{ij} = \sum_p x_i^p \overline{y}_j^p.$$

Thus we have $\chi = \sum_p \pi(x^p)\mathcal{J}\pi(y^p)\mathcal{J}^{-1}$ with $x^p = (x_i^p 1_i)_{1 \leq i \leq N}$ and $y^p = (y_i^p 1_i)_{1 \leq i \leq N}$.

Accordingly, in the complex case, the bimodule structure and the chirality of a finite spectral triple are given, up to unitary equivalence, by a symmetric and nondegenerate

Table 1

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\cap_{ij} = \mu_{ij}$	$\mu_{ij\alpha}\mu_{ij\bar\alpha} \geq 0$ $\cap_{ij} = \sum_\alpha \mu_{ij\alpha}$	$\cap_{ij} = 2\mu_{ij}$
$\mathbb{C}$	$\mu_{i\alpha j}\mu_{i\bar\alpha j} \geq 0$ $\cap_{ij} = \sum_\alpha \mu_{i\alpha j}$	$\mu_{i\alpha j}\mu_{i\bar\alpha j\bar\beta} \geq 0$ $\cap_{ij} = \sum_{\alpha\beta} \mu_{i\alpha j\beta}$	$\mu_{i\alpha j}\mu_{i\bar\alpha j} \geq 0$ $\cap_{ij} = 2\sum_\alpha \mu_{i\alpha j}$
$\mathbb{H}$	$\cap_{ij} = 2\mu_{ij}$	$\mu_{ij\alpha}\mu_{ij\bar\alpha} \geq 0$ $\cap_{ij} = 2\sum_\alpha \mu_{ij\alpha}$	$\cap_{ij} = 4\mu_{ij}$

matrix μ with entries in $\mathbb{Z}$, which is the matrix of the intersection form. Furthermore, it corresponds to a S^0-real structure if only if the intersection form is an even quadratic form. In the real case, it is still given by a symmetric matrix with entries in $\mathbb{Z}$ whose indices label irreducible representations of $\mathcal{A}$. However, it is no longer the intersection form and the signs of its entries satisfy additional requirements that are summarized, together with its relation with the intersection form, in Table 1.

In this table, the indices i and j label, for matrices with entries in a given field, the simple algebras tha appear in the decomposition of $\mathcal{A}$. In the complex case, the indices α and β distinguish a representation from its complex conjugate.

Conclusion. The bimodule structure and the chirality of a finite spectral triple are given, up to unitary equivalence, by a symmetries and nondegenerate matrix of multiplicities. The moduli of the entries of the matrix of multiplicities determine the bimodule structure and their signs yield the chirality.

3.4. Dirac operator

As we have already pointed out, a first-order operator D on a $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ is an operator that satisfies

$$D(x\psi y) = xD(\psi y) + D(x\psi)y - xD(\psi)y$$

for any $\psi \in \mathcal{M}$, $x \in \mathcal{A}$ and $y \in \mathcal{B}$. If we denote by $L_L(\mathcal{M})$ (resp. $L_R(\mathcal{M})$, $L_1(\mathcal{M})$) the space of left $\mathcal{A}$-linear operators (resp. right $\mathcal{B}$-linear operators, first-order operators), we have

$$L_L(\mathcal{M}) + L_R(\mathcal{M}) \subset L_1(\mathcal{M}).$$

If $\mathcal{M}$ is finite dimensional and if the algebras are direct sum of matrix algebras, the converse also holds

$$L_L(\mathcal{M}) + L_R(\mathcal{M}) = L_1(\mathcal{M}).$$

To prove it, we decompose the representations associated with the left and right actions into irreducible ones and determine the corresponding matrix elements of D.

In particular, this applies to the Dirac operator $\mathcal{D}$ of a finite spectral triple that can be written as

$$\mathcal{D} = \mathcal{D}_L + \mathcal{D}_R$$

with

$$[\mathcal{D}_L, \pi(A)] = [\mathcal{D}_R, \mathcal{J}\pi(A)\mathcal{J}^{-1}] = 0.$$

Moreover, since $\mathcal{D}$ anticommutes with χ, so do $\mathcal{D}_L$ and $\mathcal{D}_R$. Accordingly, the previous decomposition is unique. Indeed, if we have, with obvious notations,

$$\mathcal{D} = \mathcal{D}_L^1 + \mathcal{D}_R^1 = \mathcal{D}_L^2 + \mathcal{D}_R^2,$$

then

$$\mathcal{D}_L^1 - \mathcal{D}_L^2 = \mathcal{D}_R^2 - \mathcal{D}_R^1$$

commutes with both $\pi(\mathcal{A})$ and $\mathcal{J}\pi(\mathcal{A})\mathcal{J}^{-1}$. Thus, it commutes with χ, and since it also anticommutes with χ, it vansihes and the previous decomposition is unique. Furthermore, $\mathcal{D}$ commutes with $\mathcal{J}$, so that, using the unicity, we have

$$\mathcal{D}_L = \mathcal{J}\mathcal{D}_R\mathcal{J}^{-1}.$$

Consequently, the Dirac operator of finite spectral triple is given by

$$\mathcal{D} = \Delta + \mathcal{J}\Delta\mathcal{J}^{-1},$$

where Δ is a hermitian operator on $\mathcal{H}$, which commutes with $\mathcal{J}\pi(\mathcal{A})\mathcal{J}^{-1}$ and anticommutes with χ. For a S^0-real spectral triple, it is sufficient to take into account the particle bimodule. The Dirac operator can be written as a direct sum,

$$\mathcal{D} = \mathcal{D}_P \oplus \mathcal{D}_A,$$

where $\mathcal{D}_P$ is a first-order operator of the particle bimodule. Using the unicity, we have

$$\mathcal{D}_A = \overline{\mathcal{D}_P}.$$

The decomposition of $\mathcal{D}$ remains valid for the bimodule structure of the particle space,

$$\mathcal{D}_P = \Delta_P + \overline{\Delta}_A$$

with

$$[\Delta_A, \pi_A(\mathcal{A})] = [\Delta_P, \pi_P(\mathcal{A})] = 0.$$

Let us end this section by three remarks that are useful when this construction is applied to model building in gauge theory. First of all, we can give an explicit expression of Δ in terms of the Dirac operator. Indeed,

$$\mathcal{D} = -\int_G dg\pi(g)[\mathcal{D}, \pi(g^*)] + \int_G dg\pi(g)\mathcal{D}\pi(g^*),$$

where dg is the Haar measure on the group G of unitary elements of $\mathcal{A}$. Due to the translational invariance of the Haar measure,

$$\left[\int_G dg \pi(g) \mathcal{D} \pi(g^*), \pi(x) \right] = 0$$

for any $x \in \mathcal{A}$. Furthermore, it follows from the reality axiom and the first-order condition that

$$\left[\int_G dg \pi(g) [\mathcal{D}, \pi(g^*)], \mathcal{J} \pi(x) \mathcal{J} \right] = 0$$

for any $x \in \mathcal{A}$. Thus, it is a decomposition of the Dirac operator into left and right module homomorphisms that anticommute with χ. Using the unicity, we have

$$\Delta = - \int_G dg \pi(g) [\mathcal{D}, \pi(g^*)].$$

This expression of Δ is very useful since it means that Δ is a 1-form.

Secondly, the operator Δ satisfies very simple commutation and anticommutation relations and we can give its explicit form in terms of complex matrices. Let

$$\mathcal{H} = \bigotimes_{ij} \mathcal{H}_{ij}$$

be the decomposition of the Hilbert space where i and j label irreducible representations of $\mathcal{A}$, and $P_{ij} \mathcal{H} \rightarrow \mathcal{H}_{ij}$ be the orthogonal projection onto $\mathcal{H}_{ij}$. We define the matrix elements of Δ by

$$\Delta_{ij}^{kl} = P_{kl} \Delta P_{ij}^*.$$

The latter are given by

$$\Delta_{ij}^{kl} = \delta_{ji} M_{ik,j} \otimes I_{n_j},$$

where $M_{ik,j} \in M_{\mu_{ij} n_i \times \mu_{kj} n_k}(\mathbb{C})$ is such that $M_{ik,j} = M_{ki,j}^*$ if $\mu_{ij} \mu_{kj} < 0$, and zero otherwise. This gives us a complete description of the operator Δ parametrized by complex matrices, and we recover the Dirac operator as $\mathcal{D} = \Delta + \mathcal{J} \Delta \mathcal{J}^{-1}$.

Thirdly, notice that most of the time, the Dirac operator $\mathcal{D}$ only appears in noncommutative geometry via the commutation relation $[\mathcal{D}, \pi(x)], x \in \mathcal{A}$. Since $\mathcal{D} = \Delta + \mathcal{J} \Delta \mathcal{J}^{-1}$ and $[\mathcal{J} \Delta \mathcal{J}^{-1}, \pi(x)] = 0$ for any $x \in \mathcal{A}$, one can replace $\mathcal{D}$ by Δ in any commutation relation with an element of the algebra. This simplifies, for finite spectral triples, many computational involving the Dirac operator, such as, for instance, the computation of the metric on the space of pure states of $\mathcal{A}$ and its perturbation by inner fluctuations [7].

Finally, we use this decomposition to compute the 1-forms. We recall that the space of 1-forms $\Omega_D^1(\mathcal{A})$ is defined by

$$\Omega_{\mathcal{D}}^1(\mathcal{A}) = \left\{ \sum_p \pi(x^p)[\mathcal{D}, \pi(y^p)] \text{ with } x^p, y^p \in \mathcal{A} \right\}.$$

In particular,

$$\Delta = -\int \mathrm{d}g\pi(g)[\mathcal{D}, \pi(g^*)],$$

is a 1-form. Moreover, if $\omega \in \Omega_{\mathcal{D}}^1\mathcal{A}$ and $x, y \in \mathcal{A}$, we have $\pi(x)\omega\pi(y) \in \Omega_{\mathcal{D}}^1(\mathcal{A})$ and $\Omega_{\mathcal{D}}^1(\mathcal{A})$ is a bimodule and thus contains the bimodule generated by Δ. Conversely, if we express any element $\omega \in \Omega_{\mathcal{D}}^1(\mathcal{A})$ using commutators with $\mathcal{D}$ and then, replace $\mathcal{D}$ by Δ, we see that $\Omega_{\mathcal{D}}^1(\mathcal{A})$ is included in the bimodule generated by Δ. Accordingly, the space of 1-forms $\Omega_{\mathcal{D}}^1(\mathcal{A})$ is the bimodule generated by Δ, and this allows us, using the expression of Δ in terms of complex matrices, to determine $\Omega_{\mathcal{D}}^1(\mathcal{A})$ in Section 5.

Conclusion. A finite spectral triple is given, up to unitary equivalence by a symmetric and nondegenerate matrix of multiplicities and a hermitian operator Δ commuting with the right action and anticommuting with the chirality.

4. Classification and diagrams

In the following sections, we develop a diagrammatic approach to finite spectral triples. We first associate to any finite triple a diagram and then, we classify and construct all such triples within this approach.

4.1. Diagram associated with a spectral triples

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finite spectral triple whose matrix of multiplicities is μ. To each couple (i, j) of irreducible representations of $\mathcal{A}$ such that $\mu_{ij} \neq 0$, we associate a vertex of type $\ominus$ if $\mu_{ij} < 0$ and a vertex of type $\oplus$ if $\mu_{ij} > 0$. We relate the vertices (i, j) and (k, l) if only if the corresponding matrix element of the Dirac operator,

$$\mathcal{D}_{ij}^{kl} = P_{ij}\mathcal{D}P_{kl}^*,$$

does not vanish. According to the axioms, the diagram has the following properties.

– (i, j) is related to (k, l) if and only if (k, l) is related to (i, j) as follows from $\mathcal{D} = \mathcal{D}^*$. To simplify, we draw only one edge between two vertices.
– Since $\mathcal{D}$ and $\mathcal{J}$ commute, the diagram is symmetric with respect to its first diagonal.
– We can only relate vertices of different types, as follows from the anticommutation relation of χ and $\mathcal{D}$.
– As a consequence of the first-order condition, all edges are either vertical or horizontal. Vertical edges correspond to matrix elements of Δ and the horizontal ones to those of $\mathcal{J}\Delta\mathcal{J}^{-1}$.

For instance, let us build the diagram pertaining to the standard model. The algebra $\mathcal{A}$ is

$$\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}).$$

Since it is S^0-real, we use the notations introduced in Section 2.5. The particle Hilbert spaces $\mathcal{H}_L^P$ and $\mathcal{H}_R^P$ are spanned, for $N_f = 3$ families of fermions, by

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L,$$

and

$$(u)_R, (d)_R, (c)_R, (s)_R, (t)_R, (b)_R, (e)_R, (\mu)_R, (\tau)_R,$$

where we have omitted the color index for quarks. The corresponding antiparticles form a basis of the article spaces $\mathcal{H}_L^A$ and $\mathcal{H}_R^A$. Within these bases, the representation is given by

$$\pi_L^P(a) = \text{diag}(a \otimes I_{3N_f}, a \otimes I_{N_f}),$$
$$\pi_R^P(b) = \text{diag}(b I_{3N_f}, \overline{b} I_{3N_f}, \overline{b} I_{N_f}),$$
$$\pi_L^A(b, c) = \text{diag}(I_{2N_f} \otimes c, b I_{2N_f}),$$
$$\pi_R^A(b, c) = \text{diag}(I_{2N_f} \otimes c, b I_{N_f}),$$

where $(a, b, c) \in \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$. The mass matrix M is

$$M = \begin{pmatrix} \begin{pmatrix} M_u \otimes I_3 & 0 \\ 0 & M_d \otimes I_3 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \end{pmatrix},$$

with

$$M_u = \text{diag}(m_u, m_c, m_t),$$
$$M_d = V_{\text{CKM}} \, \text{diag}(m_d, m_s, m_b),$$
$$M_e = \text{diag}(m_e, m_\mu, m_\tau),$$

where m_p stands for the mass of particle p and V_{CKM} is the Cabibbo–Kobayashi–Maskawa mixing matrix.

Accordingly, the matrix of multiplicities is, in the basis $(\mathbb{C}, \mathbb{H}, \overline{\mathbb{C}}, M_3(\mathbb{C}))$,

$$\mu = N_f \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix},$$

and the corresponding diagram is given by Fig. 1.

For particles, vertices and edges are above the first diagonal. The three vertices $\oplus$ correspond to the singlets of isospin u_R, d_R and e_R and the two vertices $\ominus$ are associated with the leptonic and quarkonic isodoublets. The edges between them correspond to the matrix elements M_u, M_d and M_e. Since it is S^0-real, there is a complete symmetry between particles and antiparticles so that the same holds for the antiparticles. In fact, for a general S^0-real spectral triple, the matrix of multiplicities can always be written as
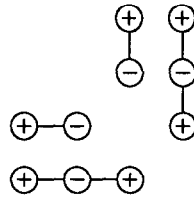
$$\mu = \epsilon + \epsilon^*.$$

Fig. 1. Diagram of the standard model.

The Dirac operator is a first-order operator of the particle bimodule and the complex con-
jugate on the antiparticle bimodule. Thus, the corresponding diagram is constructed out of
the vertices of the matrix ϵ and then symmetrized with respect to the first diagonal.

The diagram of the standard model is not connected and has four connected subdiagrams
that are intimately tied up with the "noncommutative coupling constants" x, y, $\tilde{x}$ and $\tilde{y}$,
introduced in [15]. In general, the latter is positive definite hermitian matrix Z that commutes
with $\pi(\mathcal{A})$, $\mathcal{J}\pi(\mathcal{A})\mathcal{J}^{-1}$ and $\mathcal{D}$, so that it yields a scalar product on the differential algebra.
The first two commutation relations allow us to write Z as a block diagonal matrix.

$$Z = \bigotimes_{ij} I_{n_i} \otimes Z_{ij} \otimes I_{n_j}.$$

Accordingly, to each vertex (i, j) is associated a positive definite hermitian matrix Z_{ij}. The
last commutation relation, expressed using matrix elements as

$$Z_{ij}\mathcal{D}_{ij}^{kl} = \mathcal{D}_{ij}^{kl}Z_{kl},$$

relates the matrix elements Z_{ij} associated with vertices belonging to the same connected
component.

The diagrams provide us with a natural classification of finite spectral triples. As we shall
see in the sections devoted to applications, the major feature of the underlying Yang–Mills–
Higgs model can be seen on the diagram. However, a commutative spectral triple, with the
algebra $\mathbb{C}^N$ and a noncommutative one, whose algebra is a direct sum of N simple matrix
algebras, can yield the same diagram. From the point of view of noncommutative geometry,
this is not surprising since two such algebras are Morita equivalent and thus completely
indistinguishable in Hoschild homology and K-theory. Besides, using the general construc-
tion, based on a connection ∇ on a finite projective module $\mathcal{E}$, detailed in [7], we can pass
from a commutative spectral triple to a noncommutative one which are both associated with
the same diagram.

4.2. Spectral triple associated to a diagram

Suppose we are given an algebra $\mathcal{A}$ and a symmetric matrix of multiplicities μ whose
indices label irreducible representations of $\mathcal{A}$. We require that it fulfills the constrints listed
in Table 1 and that its intersection form $\cap_{ij}$ be nondegenerate. We represent it on the plane
as the vertices of a diagram as we did in the previous section. Then we relate the vertices of

different types by vertical and horizontal lines such that the resulting diagram be symmetric with respect to its first diagonal.

To each vertical link between (i, k) and (j, k), with $i > j$, we associate a matrix $M_{ij,k} \in M_{\mu_{ik}n_i \times \mu_{j}kn_j}(\mathbb{C})$. If $j > i$, we set $M^*_{ji,k} = M^*_{ij,k}$ and $M_{ij,k} = 0$ in all other cases. The matrix elements of the operator Δ are obtained as

$$\Delta^{jl}_{ik} = \delta_{kl} M_{ij,k} \otimes I_{n_k}.$$

In particular, the diagonal elements Δ^{ij}_{ij} always vanish since we cannot relate a vertex to itself. We symmetrize this expression with respect to $\mathcal{J}$ to get the compelte Dirac operator

$$\mathcal{D} = \Delta + \mathcal{J} \Delta \mathcal{J}^{-1}$$

To obtain an explicit form of $\mathcal{D}$ in terms of complex matrices, we introduce a basis $(D^p_{ij})_{1 \leq p \leq n_i n_j}$, for example the canonical one, and we write $M_{ij,k}$ as

$$M_{ij,k} = \sum_p E^p_{ij} \otimes M^p_{ij,k}.$$

Thus, the matrix elements of the Dirac operator are

$$\mathcal{D}^{jl}_{ik} = \delta_{kl} \sum_p E^p_{ij} \otimes M^p_{ij,k} \otimes I_{n_k} + \delta_{ij} \sum_q I_{n_i} \otimes \overline{M}^p_{kl,i} \otimes \overline{E}^q_{kl}.$$

The representation, the Hilbert space and the operators $\mathcal{J}$ and χ are constructed out of the matrix of multiplicities by means of the formulae we gave throughout Sections 3.1–3.3.

5. Applications to particle models

Finite spectral triples are very simple examples of noncommutative geometries corresponding to finite spaces. The latter can be considered as the internal spaces involved in Yang–Mills–Higgs theories. More precisely, if we tensorize a finite spectral triple with the one associated to an ordinary manifold of dimension n, we obtain a noncommutative geometry of dimension n that gives rise to a Yang–Mills theory with spontaneous symmetry breaking. This section is devoted to this kind of applications. We first recall some general aspects about gauge theory in noncommutative geometry and then apply our previous classification to investigate some features of the model we obtain.

5.1. General aspects

It follows from general principles of noncommutative geometry [7], that the symmetries of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ are described by *-automorphisms of the algebra $\mathcal{A}$. among them, there are, in the noncommuttive case, inner automorphisms. The latter are defined, as $x \in \mathcal{A} \mapsto gxg^{-1} \in \mathcal{A}$, for any element g of the group G of unitary elements of $\mathcal{A}$. If the algebra is finite dimensional, the inner automorphisms are the only automorphisms

of $\mathcal{A}$ up to a discrete subgroup. Thus, they are the only possible continuous symmetries of a finite noncommutative space. To describe a symmetry of the corresponding space, the group G is represented on the Hilbert space $\mathcal{H}$ via the map $g \mapsto \pi(g)\mathcal{J}\pi(g)\mathcal{J}^{-1}$. The inner automorphisms are in complete analogy with gauge transformations.

In the framework of noncommutative geometry, $\mathcal{H}$ is the fermionic Hilbert space and $\mathcal{D}$ the Dirac operator. Accordingly, the free fermionic action is defined as

$$S_{\mathrm{F}}[\Psi] = \langle \Psi, \mathcal{D}\Psi \rangle$$

for any $\Psi \in \mathcal{H}$. Under a gauge transformation,

$$\Psi \rightarrow \pi(g)\mathcal{J}\pi(g)\mathcal{J}^{-1}\Psi,$$

the fermionic action becomes

$$\langle \Psi, (\mathcal{D} + \pi(g^*)[\mathcal{D}, \pi(g)] + \mathcal{J}\pi(g^*)[\mathcal{D}, \pi(g)]\mathcal{J}^{-1})\Psi \rangle,$$

where we have used the axioms to rewrite

$$\pi(g^*)\mathcal{J}\pi(g^*)\mathcal{J}^{-1}\mathcal{D}\pi(g)\mathcal{J}\pi(g)\mathcal{J}^{-1}$$

as

$$\mathcal{D} + \pi(g^*)[\mathcal{D}, \pi(g)] + \mathcal{J}\pi(g^*)[\mathcal{D}, \pi(g)]\mathcal{J}^{-1}.$$

To ensure gauge invariance, we introduce, as usual in field theory, a gauge field A. The latter are hermitian 1-forms and can be written as

$$A = \sum_p \pi(x^p)[\mathcal{D}, \pi(y^p)]$$

with $x^p, y^p \in \mathcal{A}$. The transformation law of A is

$$A \rightarrow \pi(g)A\pi(g^*) + \pi(g)[\mathcal{D}, \pi(g^*)]$$

so that the interacting fermionic action

$$S_F[\Psi, A] = \langle \Psi, (D + A + \mathcal{J}A\mathcal{J}^{-1})\Psi \rangle$$

is gauge invariant. Starting with the spectral action princple, "the bosonic action only depends on the spectrum of $\mathcal{D} + A + \mathcal{J}A\mathcal{J}^{-1}$", it is possible to write a bosonic action for the field A. With a suitable spectral triple, it yields, as first terms of heat kernel expansion, the bosonic sector of the standard model coupled to gravity [9].

To construct a Yang–Mills–Higgs theory, we consider a noncommutative geometry $(\mathcal{A}_t, \mathcal{H}_t, \mathcal{D}_t)$ obtained as a product of the ordinary geometry of space–time by a finite noncommutative one, described by a finite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, that corresponds to the internal degrees of freedom of the fields. $(\mathcal{A}_t, \mathcal{H}_t, \mathcal{D}_t)$ is a tensor product of spectral triples,

$$\mathcal{A}_t = C^\infty(V) \otimes \mathcal{A},$$
$$\mathcal{H}_t = L^2(V, S) \otimes \mathcal{H},$$
$$\mathcal{D}_t = \mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu) \otimes I + \gamma^5 \otimes \mathcal{D},$$

where ω_μ is the spin connection introduced to take into account the coupling of fermionic fields with gravity. Therefore, the gauge fields, A_t are given by

$$A_t = i\gamma^\mu (A_\mu + \mathcal{J} A_\mu \mathcal{J}^{-1}) + \gamma^5 \otimes (H + \mathcal{J} H \mathcal{J}^{-1}),$$

where A^μ is a vector field with values in the Lie algebra $\mathcal{G}$ of the group of unitary elements of $\mathcal{A}$, and H is a scalar field that is also a hermitian 1-form with respect to the finite geometry. The corresponding fluctuations of the metric [7] are

$$\begin{aligned} &\mathcal{D}_t + A_t + \mathcal{J}_t \mathcal{J}_t \\ &= i\gamma^\mu (\partial_\mu + \omega_\mu + A_\mu + \mathcal{J} A_\mu \mathcal{J}^{-1}) \\ &\quad + \gamma^5 \otimes (\mathcal{D} + H + \mathcal{J} H \mathcal{J}^{-1}). \end{aligned}$$

Using the decomposition $\mathcal{D} = \Delta + \mathcal{J} \Delta \mathcal{J}^{-1}$, the scalar part can be written as

$$\gamma^5 (\phi + \mathcal{J} \phi \mathcal{J}^{-1})$$

with $\phi = \Delta + H$. Moreover, since Δ is a 1-form, ϕ is a gauge field and it transforms as

$$\phi \to \pi(g) \phi \pi(g^*).$$

The interaction of fermions with scalars can be written using ϕ as

$$\langle \Psi, (\phi + \mathcal{J} \phi \mathcal{J}^{-1}) \Psi \rangle.$$

Thus, ϕ is a matrix valued scalar field that contains all Higgs fields, and the previous interaction term corresponds to Yukawa couplings.

Finally, following the spectral action principle, the Higgs potential $V(\phi)$ only depends on the spectrum of $\phi + \mathcal{J} \phi \mathcal{J}^{-1}$ and thus can be expanded as

$$V(\phi) = \sum_n \frac{a_{2n}}{(2n)!} \mathrm{Tr}(\phi + \mathcal{J} \phi \mathcal{J}^{-1})^{2n}$$

since $(\phi + \mathcal{J} \phi \mathcal{J}^{-1})^{2n+1}$ is traceless. In dimension 4, we restrict our study to polynomial potential of degree ≤ 4, so that

$$V(\phi) = a_0 + \frac{a_2}{2!} \mathrm{Tr}(\phi + \mathcal{J} \phi \mathcal{J}^{-1})^2 + \frac{a_4}{4!} \mathrm{Tr}(\phi + \mathcal{J} \phi \mathcal{J}^{-1})^4,$$

where a_0, a_2 and a_4 are real numbers related to other parameters of the theory using the spectral action principle applied to the full Dirac operator.

5.2. The gauge group

The gauge transformations correspond to the inner automorphisms of the finite dimensional algebra $\mathcal{A}$. Accordingly, the gauge group is the group G of unitary elements of $\mathcal{A}$. Since $\mathcal{A}$ is a direct sum of matrix algebras over the fields $\mathbb{R}$, C and $\mathbb{H}$, the gauge group is a direct product of the simple Lie groups $SO(n)$, $U(n)$ and $SP(n)$. As a consequence,

noncommutative geometric models can have all classical semi-simple compact Lie group as a gauge group, except those corresponding to the five exceptional Lie algebras [16].

However, the gauge group only corresponds to G up to some $U(1)$ factors. Indeed, the elements of G appear in gauge transformations as $\pi(g)\mathcal{J}\pi(g)\mathcal{J}^{-1}$ so that we have to divide G by the kernel of the representation $g \mapsto \pi(g)\mathcal{J}\pi(g)\mathcal{J}^{-1}$ to obtain the gauge group. The latter only contains $U(1)$ factors, that correspond to the $U(1)$ whose representations only appear on diagonal vertices in the diagram associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. these $U(1)$ gauge fields also disappear from the action because the action only depends on $\pi(A^\mu) + \mathcal{J}\pi(A^\mu)\mathcal{J}^{-1}$ and they are the solutions of the equation $\pi(A^\mu) + \mathcal{J}\pi(A^\mu)\mathcal{J}^{-1} = 0$.

For example, let us consider a model with fermions in the adjoint representation. It is built with $\mathcal{A} = M_n(\mathbb{C})$ and $\mathcal{H} = M_n(\mathbb{C})$. The representation of $\mathcal{A}$ on $\mathcal{H}$ is the left multiplication of matrices and $\mathcal{J}$ is given by the adjoint. Obviously, it corresponds to a diagram with a single vertex so that its gauge group is $SU(n)$ instead of $U(n) = SU(n) \times U(1)$. Moreover, from vertex uniqueness, we see immediately that it can only afford fermions with same chirality and there is no possible link, thus no Dirac operator and no Higgs field. Corresponding to the simplest diagram, this model can be considered as the minimal one. It can be extended to semi-simple algebras, with bimodule structure given by left and right multiplication. It yields a diagonal matrix of multiplicities and a diagonal diagram. Accordingly, all $U(1)$ disappear but there is still no possible links, thus no Higgs fields and no spontaneous symmetry breaking. For the standard model (see Fig. 1), the diagram has no diagonal elements. Therefore, the unwanted $U(1)$ in the group of unitary elements of $\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ does not disappear and has to be eliminated by hand.

In more sophisticated models, we often have to remove by hand some $U(1)$ factors for physical reasons such as anomaly cancellation (of course, it may also be necessary to remove non-Abelian Lie subgroups but this will change too much the noncommutative geometric setting of the model). This is achieved by means of a unimodularity condition, that can be defined as a set of linear conditions imposed at the Lie algebra level in order to eliminate the unwanted $U(1)$ factors. For instance, if we want our model to be gauge anomaly free, we impose that the generators of the Lie algebra of the gauge group satisfy the cubic relation

$$\mathrm{Tr}(T^a\{T^b, T^c\}) = 0.$$

Since the representation we deal with are rather simple (tensor products of fundamental ones or their complex conjugates), the gauge anomaly cancellation can be expressed as constraints on the matrix of multiplicities of the model. For instance, let us consider an S^0-real spectral triple with the complex algebra

$$\mathcal{A} = \bigoplus_i M_{n_i}(\mathbb{C}),$$

and a complex representation. Taking into account the occurrence of left- and right-handed fermions, the condition of anomaly is simply

$$\mathrm{Tr}_p[\chi(\pi(x) + \mathcal{J}\pi(x)\mathcal{J}^{-1})^3] = 0 \quad \text{for } x \in \mathcal{G},$$

where $\mathcal{G}$ is the Lie algebra of the gauge group and Tr_p denotes the trace restricted to particle space. To express it in terms of the matrix of multiplicites, we write any $x_i \in u(n_i)$ (Lie algebra of $U(n_i)$) as $x_i = \lambda_i + u_i$, with $\lambda_i \in su(n_i)$ and $u_i \in u(1)$. Since we only want to remove some $U(1)$ factors, we require that this condition be satisfied at least for all non-Abelian factors. It yields three different type of constraints on the matrix of multiplicities ϵ

$$\sum_j n_j(\epsilon_{ij} - \epsilon_{ji}) = 0 \quad \text{if } n_i \geq 3,$$

$$\sum_j n_j(\epsilon_{ij} - \epsilon_{ji})(u_i - u_j) = 0 \quad \text{if } n_i \geq 2,$$

$$\sum_{ij} n_i n_j(\epsilon_{ij} - \epsilon_{ji})(u_i - u_j)^3 = 0.$$

The first equations are rigid constraints that must be satisfied by the matrix of multiplicites. The second ones are linear equations for the u_i's so that they may be taken as unimodularity conditions. The last one is a cubic relation on the "hypercharges", as in the standard model. It is a well-known fact that for the standard model, anomaly cancellation is equivalent to the unimodularity condition [17]. In other words, if the second relations are satisfied, the third one also is. Obviously, this is not a general feature of the models. Moreover, for the standard model, the first relation is empty since the color sector is vectorial.

We derived these relations for complex representations of complex algebras, but their general structure remains unchanged in the real case, since unitary elements of real and quaternionic matrices are traceless as well as their cubes. We still end up with three types of constraints: some rigid ones on the matrix of multiplicities, some linear equations for the $U(1)$ factors and a cubic relation for the "hypercharges".

There is an other reason to remove an unwanted $U(1)$. Since the Hilbert space must be equipped with a representation of an associative algebra, it is not possible, in general, to build a singlet under a given gauge group. To proceed, we introduce an additional summand $\mathbb{C}$ in the algebra, so that this singlet sits in a representation of the associative algebra $\mathbb{C}$. Then, we remove the corresponding $U(1)$ factor from the gauge group and we end up with a singlet under gauge transformations.

5.3. The fermionic representation

Within the diagrammatic approach, to each couple (i, j) of irreducible representations is associated a vertex (i, j) of the diagram. This vertex corresponds to the Hilbert space $\mathcal{H}_{ij}$ appearing in the decomposition of $\mathcal{H}.\mathcal{H}_{ij}$ is endowed with a bimodule structure given by i (left action) and $\bar{j}$ (right action). The restriction of $\pi(g)\mathcal{J}\pi(g)\mathcal{J}^{-1}$ to $\mathcal{H}_{ij}$ is $g_i \otimes I_{|\mu_{ij}|} \otimes \bar{g}_j$, so that vectors of $\mathcal{H}_{ij}$ transform in the tensor product $i \otimes \bar{j}$ of the corresponding representations i and j of the simple factors of gauge group, with multiplicity $|\mu_{ij}|$. If the two factors are distinct, the tensor product is irreducible as representation of G and, for each simple factor it is just the fundamental one or its complex conjugate. Otherwise, the tensor

product is not irreducible. For $SO(n)$ and $SP(n)$, the tensor product of two fundamental representations yields the adjoint one, the symmetric and traceless one and a singlet. For $SU(n)$ the situation is different, since we can tensorize the fundamental one and its complex conjugate, and we decompose the resulting product using Young boxes. In the analysis, we do not care about the unimodularity conditions. However, the latter are simply linear conditions imposed on the $U(1)$, and do not change the general structure of the fermionic representation.

Moreover, to each vertex is associated a sign corresponding to the eigenvalue of χ on $\mathcal{H}_{ij}$. Accordingly, fermions of $\mathcal{H}_{ij}$ are chiral ones, right-handed if $\mu_{ij} > 0$ and left-handed if $\mu_{ij} < 0$. As a consequence of the symmetry of μ_{ij} and the rules of Table 1, fermions of $\mathcal{H}_{ij}$ and $\mathcal{H}_{ji}$ have the same chirality as well as those of $\mathcal{H}_{ij}$ and $\mathcal{H}_{\bar{i}\bar{j}}$ and those of $\mathcal{H}_{\bar{i}j}$ and $\mathcal{H}_{i\bar{j}}$.

The charge conjugation $\mathcal{J}$ maps $\mathcal{H}_{ij}$ onto $\mathcal{H}_{ji}$, so that, if $i \neq j$, fermions of $\mathcal{H}_{ji}$ are the antifermions of those of $\mathcal{H}_{ij}$ and $\mathcal{H}_{ij}$ does not contain Majorana particles. Otherwise, $\mathcal{H}_{ii}$ is stable under charge conjugation, and there may be Majorana fermions. In general, there always are Majorana fermions in $\mathcal{H}_{ii}$, that transforms as the tensor product $i \otimes \bar{i}$, except when the spectral triple is S^0-real.

Many general properties of the fermionic sector, such as chirality, charge conjugation and transformation laws can be read on the diagram. As an example, let us try to build the $SU(5)$ grand unified model. To proceed. we need two representations of $SU(5)$: the $\bar{5}$ for right-handed fermions and the 10 for left-handed ones. The $\bar{5}$ can be obtained with the algebra $M_5(\mathbb{C}) \oplus \mathbb{C}$ and the corresponding bimodule structure is given by the couples $(\bar{5}, 1)$ or $(\bar{5}, \bar{1})$. At the gauge group level, we remove the unwanted U(1) with a unimodularity condition, as well as the one arising from $M_5(\mathbb{C})$. If we tensorize the 5 by itself we get the 10 and the 15. Thus, if we remove the fermions sitting in the 15, it works perfectly well from the point of view of representation theory. In the basis composed by the representations $M_5(\mathbb{C})$, $\overline{M_5(\mathbb{C})}$ and $\mathbb{C}$, the matrix of multiplicities reads, with N_f generations,

$$\mu = N_f \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The corresponding diagram is given in Fig. 2.

To end this section, let us note that it is impossible to build the $SO(10)$ model out of a spectral triple. Indeed, it requires the 16 (spinorial representation of $SO(10)$) that cannot be obtained as a subrepresentation of a tensor product of the fundamental one or its complex conjugate.
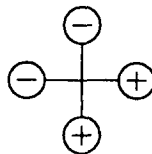

Fig. 2. Diagram of grand unified $SU(5)$ model.

5.4. The scalar fields and their representations

To determine the Higgs fields and their transformation laws, we first have to compute explicitly the space of 1-forms. We recall that Δ is a 1-form and that the space of 1-forms $\Omega^1 - \mathcal{D}(\mathcal{A})$ is the bimodule generated by Δ. Thus, if $(E_{ij}^p)_{1 \le p \le n_i n_j}$ is a basis of $M_{n_i \times n_j}(\mathbb{C})$, the 1-forms

$$\pi(E_{ii}^p \Delta \pi(E_{jj}^p)$$

span the space $\Omega_D^1(\mathcal{A})$.

However, they are not, in general, linearly independent. To find a basis of $\Omega_D^1(\mathcal{A})$, we must have a closer look at the matrix structure of Δ. To simplify our discussion, let us assume that we are dealing with a complex spectral triple. In this case, it is useful to choose $(E_{ij}^p)_{1 \le p \le n_i n_j}$ as the canonical basis of $M_{n_i \times n_j}(\mathbb{C})$. We denote it by E_{ij}^{ab}, and the multiplication rule is simply

$$E_{ij}^{ab} E_{kl}^{cd} = \delta_{jk} \delta_{bc} E_{il}^{ab}.$$

As in Section 4.2, we rewrite the matrix elements of Δ as

$$\Delta_{ik}^{jl} = \delta_{kl} \sum_p E_{ij}^{ab} \otimes M_{ij,k}^{ab} \otimes I_{n_k}.$$

Therefore,

$$\pi(E_{ii}^{ab}) \Delta \pi(E_{jj}^{cd} = \sum_k P_{ik}^* (E_{ij}^{ad} \otimes M_{ij,k}^{bc} \otimes I_{nk}) P_{jk}.$$

To construct a basis, let us consider the matrices $(M_{ij,k}^{ab})_{1 \le k \le N}$ as a column vector M_{ij}^{ab}. The vectors M_{ij}^{ab} span, when a and b vary, a vector space V_{ij} whose dimension is denoted by r_{ij}. Out of the vectors (M_{ij}^{ab}), we choose a basis of V_{ij} that we denote by $(M_{ij}^p)_{1 \le p \le r_{ij}}$. The components $M_{ij,k}^p$ of M_{ij}^p yield a basis $\theta_{ij}^{ab,p}$ of $\Omega_D^1(\mathcal{A})$ defined by

$$\theta_{ij}^{ab,p} = \sum_k P_{ik}^* (E_{ij}^{ab} \otimes M_{ij,k}^p \otimes I_{n_k}) P_{jk}.$$

Accordingly, $\Omega_1^{\mathcal{D}}(\mathcal{A})$ is a vector space of dimension $\sum_{ij} n_i r_{ij} n_j$, and it can be parametrized by complex matrices as

$$\Omega_D^1(\mathcal{A}) = \left\{ \sum_{ijkp} P_{ik}^* (\theta_{ij}^p \otimes M_{ij,k}^p \otimes I_{n_k}) P_{jk} | \theta_{ij}^p \in M_{n_i \times n_j}(\mathbb{C}) \right\}.$$

In the real case, it works in much the same way if i and j label irreducible representations of $\mathcal{A}$. However, in the quaternionic case we must tensorize the elementary matrices with Pauli matrices. We also have some additional constraints between the matrices θ_{ij} such as $\theta_{\overline{ij}} = \bar{\theta}_{ij}$, where $\bar{i}$ denotes the representation complex conjugate of i and $\bar{\theta}$ is either the complex or the $SU(2)$ conjugation.

This applies in particular for the Higgs field ϕ that can be parametrized by matrices ϕ_{ij}^p as

$$\phi = \sum_{ijkp} P_{ik}^* (\phi_{ij}^p \otimes M_{ij,k}^p \otimes I_{n_k}) P_{jk}.$$

Since ϕ is hermitian, we have

$$(\phi_{ij}^p)^* = \phi_{ji}^p.$$

Under a gauge transformation,

$$\phi \to \pi(g) \phi \pi(g^{-1}),$$

the matrix valued scalar fields ϕ_{ij}^p transform as

$$\phi_{ij}^p \to g_i \phi_{ij}^p g_j^{-1},$$

where, as usual, g_i denotes the image of g through the irreducible representation labeled by i. Accordingly, the fields ϕ_{ij}^p transforms as the tensor product $i \otimes \bar{j}$ of the two irreducible representations i and $\bar{j}$.

On the diagram, the fields ϕ_{ij}^p correspond to vertical links between the rows i and j. Since such links cannot occur within the same row, the Higgs fields do not transform as the tensor product $i \otimes \bar{i}$, but all other tensor products are allowed. In particular, we never end up with Higgs fields in the adjoint representation. Therefore, it is not possible to build the conventional $SU(5)$ grand unified model in our framework. Although we proved that it can work at the level of the fermionic representation, to break $SU(5)$ down to $SU(2) \times U(1) \times SU(3)$, we need some Higgs fields sitting in the adjoint representation of $SU(5)$.

The Yukawa coupling between ϕ and Ψ,

$$\langle \Psi, (\phi + \mathcal{J} \phi \mathcal{J}^{-1}) \Psi \rangle,$$

can be expressed in terms of the fields ϕ_{ij}^p if we expand Ψ over the tensor products we introduced in Section 3.2. As a results, it appears the Yukawa couplings between the chiral fermions of the vertices (i, k) and (j, l) and the scalar fields ϕ_{ij} or $\bar{\phi}_{kl}$ correspond to the vertical or horizontal edges that can occur between the vertices (i, k) and (j, l).

According to the spectral action principle, the Higgs potential is a linear combination of terms like

$$\text{Tr}(\phi + \mathcal{J} \phi \mathcal{J}^{-1})^n.$$

Using the explicit form of ϕ in terms of the Higgs fields ϕ_{ij}^q, the previous function can be expanded over monomials like

$$\text{Tr}(\phi_{i_1 i_2}^{p_1} \dots \phi_{i_r i_1}^{p_r}) \quad \text{Tr}(\phi_{j_1 j_2}^{q_1} \dots \phi_{j_s j_1}^{q_s})$$

with $r + s = n$. the coefficient of this monomial in the Higgs potential depends on the matrices $M_{ij,k}^p$. It corresponds canonically to a loop of length n contained in the diagram,

made of r horizontal and s vertical edges. When we are dealing with loops, we must be aware that we have actually two links between the vertices (i, k) and (j, l), one for $M_{ij,k}$ and the other for $M_{ji,k}$, even if, for simplicity, we represent only one. For instance, the sequence of vertices,

$$(i, k) \to (j, k) \to (i, k),$$

must be considered as a loop. We refer to this kind of loops, obtained by going forwards and backwards along a tree, as trivial loops. The previous example corresponds, at the level of the Higgs potential, to the monomial

$$\text{Tr}(\phi_{ij}\phi_{ji}) = \text{Tr}(\phi_{ij}\phi_{ij}^*),$$

which is a mass term. In general, mass terms only corresponds to trivial loops. Nontrivial ones, such as the sequence

$$(i, k) \to (j, k) \to (j, l) \to (i, l) \to (i, k),$$

with $i \neq j$ and $k \neq l$, can only occur for $n \geq 4$, i.e., for interacting terms.

6. Conclusion

Starting with the notion of finite spectral triple derived from general axioms, we give all possible noncommutative geometries on finite spaces. The latter are given, up to unitary equivalence, by a symmetric and nondegenerate matrix of multiplicities μ with entires in $\mathbb{Z}$ and an operator Δ. The underlying bimodule structure is encoded in the modules of the entries of the matrix of multiplicities whereas their signs correspond to the chirality. The matrix is symmetric if only if it corresponds to a real structure, and it follows from Poincaré duality that it is nondegenerate. Finite spectral triples are classified and constructed using diagrams whose vertices correspond to their matrix of multiplicities and whose edges represent the structure of the Dirac operator.

Finite spectral triples are also tools to build some Yang–Mills–Higgs models, and a general feature of the resulting models can be summarized as follows:

– The gauge group is, up to some $U(1)$ factors, a direct product of the simple Lie groups $SO(n)$ and $SP(n)$.
– Fermions transform as tensor products of two fundamental representations (or their complex conjugate) of the simple factors. Multiplets of chiral fermions correspond to the vertices of a diagram labeled by couples of fundamental representation or their complex conjugate.
– The edges of the previous diagram correspond naturally to Yukawa couplings. The resulting Higgs fields also transform as tensor products of two fundamental representations (or their complex conjugate) of the simple factors of the gauge group, except those leading to the adjoint one.
– The Higgs potential, computed with the spectral action principle, can be expanded over monomials that correspond to loops of the diagram.

This study confirms, in a more general setting, some earlier results pertaining to the standard model [18], its $U(1)$ extensions [19] and grand unified theories [20]. Furthermore, such a classification has also been worked out independently by Paschke and Sitarz in [21], putting emphasis on Hopf algebra symmetries.

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